Simple Derivation of Markov-correlated Bernoulli Trial Probability Generating Function

Preprint · March 2008

DOI: 10.13140/RG.2.2.21195.64800

1 author:

Yasunari Tosa
Stinger Ghaffarian Technologies Inc.

Simple Derivation of Markov-correlated Bernoulli Trial Probability Generating Function

Yasunari Tosa
Unpublished work (March 2008) at Retica Systems Inc., Waltham, MA 02451

Abstract

We derive the closed expression for the Markov-correlated Bernoulli trial probability generating function and its mean value for arbitrary N.

KEYWORDS

Probability generating function; Bernoulli trial; Markov chain.

I. Introduction

The Markov-correlated Bernoulli trials is the most simplest model of correlated Bernoulli trials. It was dealt by Viveros, Balasubramanian, and Balakrishnan (1994). Unfortunately their result for the generating functions and its mean were not correct (this can be verified by trying their results for $p_1 = p_2$. We will derive the correct results in this paper.

II. Probability Generating Function

We derive that the Markov-correlated Bernoulli trial probability generating function (PGF) $\varphi_N(t) = E(t^{X_N}) = \sum_{k=0}^{N} \mathrm{P}(\mathrm{X_N} = k)\, t^k \; (N \geq 2)$ where X_N denotes the number of successes in the first N trials is given by

$$\varphi_N(t) = \frac{(\alpha^N - \beta^N)p_0 t + (\alpha^{N-1} - \beta^{N-1})((q_1 - q_2)p_0 t + (p_2 t + q_2)q_0) - (\alpha^{N-2} - \beta^{N-2})\alpha\beta q_0}{(\alpha - \beta)}$$

For N=1, it is trivial and $\varphi_1(t) = p_0\, t + q_0$. The parameters, α and β, are eigenvalues of the modified Markov matrix

$$\begin{pmatrix} p_1 t & p_2\, t \\ q_1 & q_2 \end{pmatrix}$$

They satisfy

$$\alpha + \beta = p_1\, t + q_2$$

$$\alpha\beta = (p_1 q_2 - p_2 q_1)t = (p_1 - p_2)t$$

The Markov chain parameters, p_0, p_1, p_2, are defined to be $p_0 = \mathrm{P}(O_1 = S)$, $p_1 = \mathrm{P}(O_i = S \mid O_{i-1} = S)$, $p_2 = \mathrm{P}(O_i = S \mid O_{i-1} = F)$, where O_i is the outcome of trial i $(i \geq 1)$ and S and F denote success and failure, respectively. Let $q_j = 1 - p_j$ $(j = 0,1,2)$.

Regarding the numerator as a function of α and β, i.e. $f(\alpha,\beta) = \varphi_N(t)(\alpha - \beta)$, you find $f(\alpha,\alpha) = 0$ and $f(\beta,\alpha) = f(\alpha,\beta)$. Thus the numerator has the factor $(\alpha - \beta)$ and is a symmetric polynomial of α and β. That is, the numerator has the form of $(\alpha - \beta)g(\alpha + \beta, \alpha\beta)$. Therefore, as expected, $\varphi_N(t)$ is a polynomial of t up to the power of N, since $\alpha + \beta$ and $\alpha\beta$ are linear in t and the highest power of $\alpha + \beta$ is $N - 1$, which is multiplied by $p_0 t$.

By knowing this expression explicitly, we can calculate the mean $E(X_N) = \varphi_N'(1)$, and the variance $V(X_N) = E(X_N^2) - E(X_N)^2 = \varphi_N''(1) + \varphi_N'(1) - (\varphi_N'(1))^2$, using the fact that $\varphi_N'(t) = E(X_N\, t^{X_N - 1})$ and $\varphi_N''(t) = E(X_N(X_N - 1)t^{X_N - 2})$.

Note that when $p_1 = p_2 = p$, we have $\beta = 0$, which leads to

$$\varphi_N(t) = (p\, t + q)^{N-1}(p_0 t + q_0)$$

When $p = p_1 = p_2 = p_0$, this leads the standard Bernoulli trial PGM as expected:

$$\varphi_N(t) = (pt + q)^N$$

The starting point is to decompose $\varphi_N(t)$ as (introduced by Viveros, Balasubramanian, and Balakrishnan (1994)):

$$\varphi_N(t) = \varphi_N^{(S)} + \varphi_N^{(F)}$$

where $\varphi_N^{(S)}$ and $\varphi_N^{(F)}$ are the net contributions to the PGF from N sequences of trials ending in a success and in a failure, respectively. Then we have their difference equation:

$$\begin{pmatrix} \varphi_N^{(S)} \\ \varphi_N^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}^{(S)} \\ \varphi_{N-1}^{(F)} \end{pmatrix}$$

Their result Eq. (20) can be shown to be wrong easily when you compare their result for $p_1 = p_2$. Since it is easy to calculate N=1 case, this equation is reduced to

$$\begin{pmatrix} \varphi_N^{(S)} \\ \varphi_N^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix}^{N-1} \begin{pmatrix} p_0\, t \\ q_0 \end{pmatrix}$$

The result of this paper is to explicitly solve this equation.

One can convince oneself for the validity of this equation by explicitly calculating $\varphi_N^{(S)}$ and $\varphi_N^{(F)}$ by hand following Markov chain, up to n=4:

$\varphi_1^{(S)} = p_0\, t$

$$\varphi_1^{(F)} = q_0$$

$$\varphi_2^{(S)} = p_2\, q_0\, t + p_1\, p_0\, t^2$$

$$\varphi_2^{(F)} = q_2\, q_0 + q_1\, p_0\, t$$

$$\varphi_3^{(S)} = p_2\, q_2\, q_0\, t + (p_2\, q_1\, p_0 + p_1\, p_2\, q_0)\, t^2 + p_1\, p_1\, p_0\, t^3$$

$$\varphi_3^{(F)} = q_2\, q_2\, q_0 + (\, q_2\, q_1\, p_0 + q_1\, p_2\, q_0\,)t + q_1\, p_1\, p_0\, t^2$$

$$\varphi_4^{(S)} = p_2\, q_2\, q_2\, q_0\, t + (p_1\ p_2\, q_2\, q_0 + p_2\, q_1\, p_2\, q_0 + p_2\, q_2\, q_1\, p_0)\, t^2$$
$$+ (p_1\, p_1\, p_2\, q_0 + p_1\, p_2\, q_1\, p_0 + p_2\, q_1\, p_1\, p_0\,)\, t^3 + p_1 p_1 p_1 p_0\, t^4$$

$$\varphi_4^{(F)} = q_2\ q_2\, q_2\, q_0 + (\, q_2\, q_2\, q_1\, p_0 + q_2\, q_1\, p_2\, q_0 + q_1\, p_2\, q_2\, q_0)t + q_1\, p_1\, p_1\, p_0\ \ t^2$$

…

You can verify the difference equation for n=2:

$$\begin{pmatrix} \varphi_2^{(S)} \\ \varphi_2^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} p_0\, t \\ q_0 \end{pmatrix}$$

For n=3:

$$\begin{pmatrix} \varphi_3^{(S)} \\ \varphi_3^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \varphi_2^{(S)} \\ \varphi_2^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix}^2 \begin{pmatrix} p_0\, t \\ q_0 \end{pmatrix}$$
$$= \begin{pmatrix} p_1 p_1\, t^2 + p_2\, q_1\, t & p_1\, p_2\, t^2 + p_2\, q_2\, t \\ q_1\, p_1\, t + q_2\, q_1 & q_1\, p_2\, t + q_2\, q_2 \end{pmatrix} \begin{pmatrix} p_0\, t \\ q_0 \end{pmatrix}$$
$$= \begin{pmatrix} p_1\, p_1\, p_0\, t^3 + p_2\, q_1\, p_0\, t^2 + p_1\, p_2\, q_0\, t^2 + p_2\, q_2\, q_0\, t \\ q_1\, p_1\, p_0\, t^2 + q_2\, q_1\, p_0\, t + q_1\, p_2\, q_0\, t + q_2\, q_2\, q_0 \end{pmatrix}$$

The task is to obtain the general expression for $\begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix}^{N-1}$. For any real matrix A whose eigenvalues are **not degenerate**, we can find a normal matrix D such that $D^{-1} A\, D$ is diagonal and D can be constructed from eigenvectors. For our case, find eigenvectors using the following equation:

$$\begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \rho \begin{pmatrix} u \\ v \end{pmatrix}$$

where ρ is the eigenvalue of the relevant matrix which satisfies

$$\rho^2 - (\, p_1 t + q_2)\rho + (p_1\, q_2 - q_1 p_2)t = 0$$

Thus the diagonalization matrix is given with α and β by (normalization being irrelevant):

$$D = \begin{pmatrix} 1 & 1 \\ \dfrac{q_1}{\alpha - q_2} & \dfrac{q_1}{\beta - q_2} \end{pmatrix}$$

Its inverse is given by

$$D^{-1} = \begin{pmatrix} \dfrac{(\alpha - q_2)}{(\alpha - \beta)} & -\dfrac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \\ -\dfrac{(\beta - q_2)}{(\alpha - \beta)} & \dfrac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \end{pmatrix}$$

The denominator $(\alpha - \beta)$ signifies the non-degeneracy requirement for eigenvalues.

We can verify the following

$$D^{-1} \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Instead, we verify the next one:

$$\begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} = D \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} D^{-1}$$

or

$$\begin{pmatrix} 1 & 1 \\ \frac{q_1}{\alpha - q_2} & \frac{q_1}{\beta - q_2} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \frac{(\alpha - q_2)}{(\alpha - \beta)} & -\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \\ -\frac{(\beta - q_2)}{(\alpha - \beta)} & \frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \end{pmatrix} =$$

$$\begin{pmatrix} \alpha & \beta \\ \frac{\alpha\, q_1}{(\alpha - q_2)} & \frac{\beta q_1}{(\beta - q_2)} \end{pmatrix} \begin{pmatrix} \frac{(\alpha - q_2)}{(\alpha - \beta)} & -\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \\ -\frac{(\beta - q_2)}{(\alpha - \beta)} & \frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} \end{pmatrix}$$

The 11 term is

$$\frac{\alpha\,(\alpha - q_2)}{(\alpha - \beta)} - \frac{\beta(\beta - q_2)}{(\alpha - \beta)} = \frac{(\alpha - \beta)(\alpha + \beta) - (\alpha - \beta)q_2}{(\alpha - \beta)} = (\alpha + \beta) - q_2 = (p_1 t + q_2) - q_2 = p_1\, t$$

The 21 term is

$$\frac{\alpha\, q_1}{(\alpha - q_2)} \frac{(\alpha - q_2)}{(\alpha - \beta)} - \frac{\beta q_1}{(\beta - q_2)} \frac{(\beta - q_2)}{(\alpha - \beta)} = \frac{\alpha\, q_1}{(\alpha - \beta)} - \frac{\beta q_1}{(\alpha - \beta)} = q_1$$

The 12 term is

$$- \frac{\alpha\,(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} + \frac{\beta(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta)q_1} = - \frac{(\alpha - q_2)(\beta - q_2)}{q_1} = - \frac{\alpha\beta - q_2(\alpha + \beta) + q_2^2}{q_1} =$$

$$- \frac{(p_1\, q_2 - q_1\, p_2)t - q_2\,(p_1\, t + q_2) + q_2^2}{q_1} = p_2\, t$$

The 22 term is

$$-\frac{\alpha\, q_1}{(\alpha - q_2)}\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1} + \frac{\beta q_1}{(\beta - q_2)}\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1} = -\frac{\alpha\,(\beta - q_2)}{(\alpha - \beta)} + \frac{\beta(\alpha - q_2)}{(\alpha - \beta)} = q_2$$

Finally we solved the equation as

$$\begin{pmatrix}\varphi_N^{(S)}\\ \varphi_N^{(F)}\end{pmatrix} = \begin{pmatrix}p_1\, t & p_2\, t\\ q_1 & q_2\end{pmatrix}^{N-1}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix} = \left(D\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}D^{-1}\right)^{N-1}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix} = D\begin{pmatrix}\alpha^{N-1} & 0\\ 0 & \beta^{N-1}\end{pmatrix}D^{-1}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix}$$

Or

$$\begin{pmatrix}\varphi_N^{(S)}\\ \varphi_N^{(F)}\end{pmatrix} = \begin{pmatrix}1 & 1\\ \frac{q_1}{\alpha - q_2} & \frac{q_1}{\beta - q_2}\end{pmatrix}\begin{pmatrix}\alpha^{N-1} & 0\\ 0 & \beta^{N-1}\end{pmatrix}\begin{pmatrix}\frac{(\alpha - q_2)}{(\alpha - \beta)} & -\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1}\\ -\frac{(\beta - q_2)}{(\alpha - \beta)} & \frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1}\end{pmatrix}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix}$$

$$= \begin{pmatrix}\alpha^{N-1}\frac{(\alpha - q_2)}{(\alpha - \beta)} - \beta^{N-1}\frac{(\beta - q_2)}{(\alpha - \beta)} & -\alpha^{N-1}\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1} + \beta^{N-1}\frac{(\alpha - q_2)(\beta - q_2)}{(\alpha - \beta) q_1}\\ \alpha^{N-1}\frac{q_1}{(\alpha - \beta)} - \beta^{N-1}\frac{q_1}{(\alpha - \beta)} & -\alpha^{N-1}\frac{(\beta - q_2)}{(\alpha - \beta)} + \beta^{N-1}\frac{(\alpha - q_2)}{(\alpha - \beta)}\end{pmatrix}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix}$$

Noting that

$$p_2\, t = -\frac{(\alpha - q_2)(\beta - q_2)}{q_1}$$

the previous equation can be written as

$$= \begin{pmatrix}\alpha^{N-1}\frac{(\alpha - q_2)}{(\alpha - \beta)} - \beta^{N-1}\frac{(\beta - q_2)}{(\alpha - \beta)} & \alpha^{N-1}\frac{p_2\, t}{(\alpha - \beta)} - \beta^{N-1}\frac{p_2\, t}{(\alpha - \beta)}\\ \alpha^{N-1}\frac{q_1}{(\alpha - \beta)} - \beta^{N-1}\frac{q_1}{(\alpha - \beta)} & -\alpha^{N-1}\frac{(\beta - q_2)}{(\alpha - \beta)} + \beta^{N-1}\frac{(\alpha - q_2)}{(\alpha - \beta)}\end{pmatrix}\begin{pmatrix}p_0\, t\\ q_0\end{pmatrix}$$

$$\begin{pmatrix}\frac{\alpha^{N-1}}{(\alpha - \beta)}\left((\alpha - q_2)p_0\, t + p_2\, q_0\, t\right) - \frac{\beta^{N-1}}{(\alpha - \beta)}\left((\beta - q_2)p_0\, t + p_2\, q_0\, t\right)\\ \frac{\alpha^{N-1}}{(\alpha - \beta)}(q_1 p_0\, t - (\beta - q_2)q_0) - \frac{\beta^{N-1}}{(\alpha - \beta)}(q_1 p_0\, t - (\alpha - q_2)q_0)\end{pmatrix}$$

The final result for $\varphi_{N=}\ \varphi_N^{(S)} + \varphi_N^{(F)}$ is given by

$$\varphi_N = \frac{\alpha^{N-1}}{(\alpha - \beta)}\left((\alpha - q_2)p_0\, t + p_2\, q_0\, t + q_1 p_0\, t - (\beta - q_2)q_0\right)$$

$$-\frac{\beta^{N-1}}{(\alpha-\beta)}((\beta - q_2)p_0\, t + p_2\, q_0\, t + q_1 p_0\, t - (\alpha - q_2)q_0)$$

Or

$$\varphi_N = \frac{\alpha^{N-1}}{(\alpha-\beta)}\big((\alpha + q_1 - q_2)p_0\, t - (\beta - p_2\, t - q_2)q_0\big)$$

$$-\frac{\beta^{N-1}}{(\alpha-\beta)}((\beta + q_1 - q_2)p_0\, t - (\alpha - p_2\, t - q_2)q_0)$$

Or

$$\varphi_N = \frac{(\alpha^N - \beta^N)p_0 t + (\alpha^{N-1} - \beta^{N-1})((q_1 - q_2)p_0 t + (p_2 t + q_2)q_0) - (\alpha^{N-2} - \beta^{N-2})\alpha\beta q_0}{(\alpha-\beta)}$$

We can verify the validity of the result for some N's.

For N=2, we have

$$\varphi_2\ (\alpha - \beta) = (\alpha^2 - \beta^2)p_0 t + (\alpha - \beta)\big((q_1 - q_2)p_0\, t + (p_2 t + q_2)q_0\big)$$

That is,

$$\varphi_2 = (p_1\, t + q_2)\, p_0\, t + (q_1 - q_2)p_0\, t + (p_2\, t + q_2)\, q_0 = p_1 p_0\, t^2 + p_2\, q_0\, t + q_1 p_0\, t + q_2 q_0$$

For N=3, we have

$$\varphi_3\ (\alpha - \beta) = (\alpha^3 - \beta^3)p_0 t + (\alpha^2 - \beta^2)\big((q_1 - q_2)p_0\, t + (p_2\, t + q_2)q_0\big) - (\alpha - \beta)\alpha\beta q_0$$

That is,

$$\varphi_3 = ((\alpha + \beta)^2 - \alpha\beta)p_0\, t + (\alpha + \beta)\big((q_1 - q_2)p_0\, t + (p_2\, t + q_2)q_0\big) - \alpha\beta q_0$$

Noting that

$$\alpha + \beta = p_1\, t + q_2$$

$$\alpha\beta = (p_1 q_2 - p_2\, q_1)t$$

we have

$$\varphi_3 = (\alpha + \beta)^2\, p_0\, t + (\alpha + \beta)\big((q_1 - q_2)p_0\, t + (p_2\, t + q_2)q_0\big) - \alpha\beta(p_0 t + q_0)$$

Or

$$\varphi_3 = (p_1\, t + q_2)^2\, p_0\, t + (p_1\, t + q_2)\big((q_1 - q_2)p_0\, t + (p_2\, t + q_2)q_0\big) - (p_1 q_2 - p_2 q_1)(p_0 t + q_0)t$$

Or

$$\varphi_3 = p_1 p_1 p_0\, t^3 + (q_1 p_1 p_0 + p_1 p_2 q_0 + p_2 q_1 p_0) t^2 + (q_2 q_1 p_0 + q_1 p_2 q_0 + p_2 q_2 q_0) t + q_2 q_2 q_0$$

III. Mean value calculation

We derive the following for the mean:

$$\varphi_N'(1) = p_0 + (N-1)p_2 + \gamma \frac{(N-1)(1-\gamma) - (1-\gamma^{N-1})}{(1-\gamma)^2} p_2 + \gamma \frac{1-\gamma^{N-1}}{1-\gamma} p_0$$

where $\gamma = p_1 - p_2$. Note that the numerator of the third term contains the factor $(1-\gamma)^2$, since

$$(N-1)(1-\gamma) - (1-\gamma^{N-1}) = (1-\gamma)((N-1) - (1 + \gamma + \gamma^2 + \ldots + \gamma^{N-2}))$$

We now like to calculate the mean, $\varphi_N'(1)$. Note that

$$\begin{pmatrix} \varphi_N'(t) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_N^{(S)} \\ \varphi_N^{(F)} \end{pmatrix}$$

Using

$$\begin{pmatrix} \varphi_N^{(S)} \\ \varphi_N^{(F)} \end{pmatrix} = \begin{pmatrix} p_1\, t & p_2\, t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}^{(S)} \\ \varphi_{N-1}^{(F)} \end{pmatrix}$$

we have

$$\begin{pmatrix} \varphi_N'^{(S)} \\ \varphi_N'^{(F)} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}^{(S)} \\ \varphi_{N-1}^{(F)} \end{pmatrix} + \begin{pmatrix} p_1 t & p_2 t \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}'^{(S)} \\ \varphi_{N-1}'^{(F)} \end{pmatrix}$$

Thus, we now have

$$\begin{pmatrix} \varphi_N'(t) \\ 0 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}^{(S)} \\ \varphi_{N-1}^{(F)} \end{pmatrix} + \begin{pmatrix} p_1 t + q_1 & p_2 t + q_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{N-1}'^{(S)} \\ \varphi_{N-1}'^{(F)} \end{pmatrix}$$

with $\begin{pmatrix} \varphi_1^{(S)} \\ \varphi_1^{(F)} \end{pmatrix} = \begin{pmatrix} p_0 t \\ q_0 \end{pmatrix}$ *and* $\begin{pmatrix} \varphi_1'^{(S)} \\ \varphi_1'^{(F)} \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix}$. Therefore, we have for $t = 1$

$$\begin{pmatrix} \varphi_N'(1) \\ 0 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{N-2} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} + \begin{pmatrix} p_1 + q_1 & p_2 + q_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi'^{(S)}_{N-1} \\ \varphi_{N-1}'^{(F)} \end{pmatrix}$$

Or noting that $p_1 + q_1 = 1$ and $p_2 + q_2 = 1$,

$$\begin{pmatrix} \varphi_N'(1) \\ 0 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^{N-2} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} + \begin{pmatrix} \varphi_{N-1}' \\ 0 \end{pmatrix}$$

This equation is valid even for N= 2, since

$$\varphi_2(t) = p_1 p_0 t^2 + q_1 p_0 t + p_2 q_0 t + q_2 q_0$$

$$\varphi_1(t) = p_0 t + q_0$$

lead to

$$\varphi_2'(1) = 2p_1p_0 + p_2q_0 + q_1p_0 = p_1p_0 + p_2q_0 + (p_1 + q_1)p_0 = (p_1p_0 + p_2q_0) + p_0$$

Therefore

$$\begin{pmatrix} \varphi_N'(1) \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} \left\{ \sum_{j=0}^{N-2} \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}^j \right\} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

Eigenvalues for $\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$ are $p_1 - p_2$ and 1:

$$\rho^2 - (p_1 + q_2)\rho + (p_1q_2 - p_2q_1) = 0$$

$$\rho^2 - (p_1 - p_2 + 1)\rho + (p_1 - p_2) = 0$$

$$(\rho - 1)\big(\rho - (p_1 - p_2)\big) = 0$$

Thus it can be diagonalized by

$$D = \begin{pmatrix} 1 & 1 \\ -1 & \frac{q_1}{p_2} \end{pmatrix}$$

And

$$D^{-1} = \frac{1}{(p_2 + q_1)} \begin{pmatrix} q_1 & -p_2 \\ p_2 & p_2 \end{pmatrix}$$

That is,

$$\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = D \begin{pmatrix} p_1 - p_2 & 0 \\ 0 & 1 \end{pmatrix} D^{-1} = \frac{1}{p_2(p_2 + q_1)} \begin{pmatrix} p_2 & p_2 \\ -p_2 & q_1 \end{pmatrix} \begin{pmatrix} p_1 - p_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 & -p_2 \\ p_2 & p_2 \end{pmatrix}$$

Consequently

$$\begin{pmatrix} \varphi_N'(1) \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix} D \left\{ \sum_{j=0}^{N-2} \begin{pmatrix} p_1 - p_2 & 0 \\ 0 & 1 \end{pmatrix}^j \right\} D^{-1} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} \varphi_N'(1) \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} + \begin{pmatrix} p_1 - p_2 & 1 \\ 0 & 0 \end{pmatrix} \left\{ \sum_{j=0}^{N-2} \begin{pmatrix} (p_1 - p_2)^j & 0 \\ 0 & 1 \end{pmatrix} \right\} D^{-1} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix}\varphi_N'(1)\\0\end{pmatrix} = \begin{pmatrix}p_0\\0\end{pmatrix} + \begin{pmatrix}p_1-p_2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{1-(p_1-p_2)^{N-1}}{1-(p_1-p_2)} & 0\\0 & N-1\end{pmatrix}\frac{1}{1-(p_1-p_2)}\begin{pmatrix}q_1 & -p_2\\p_2 & p_2\end{pmatrix}\begin{pmatrix}p_0\\q_0\end{pmatrix}$$

$$\begin{pmatrix}\varphi_N'(1)\\0\end{pmatrix}$$
$$= \begin{pmatrix}p_0\\0\end{pmatrix} + \begin{pmatrix}p_1-p_2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}\frac{1-(p_1-p_2)^{N-1}}{1-(p_1-p_2)} & 0\\0 & N-1\end{pmatrix}\frac{1}{1-(p_1-p_2)}\begin{pmatrix}\left(1-(p_1-p_2)\right)p_0 - p_2\\p_2\end{pmatrix}$$

$$\begin{pmatrix}\varphi_N'(1)\\0\end{pmatrix} = \begin{pmatrix}p_0\\0\end{pmatrix} + \begin{pmatrix}\frac{1-(p_1-p_2)^{N-1}}{1-(p_1-p_2)}(p_1-p_2) & N-1\\0 & 0\end{pmatrix}\frac{1}{1-(p_1-p_2)}\begin{pmatrix}\left(1-(p_1-p_2)\right)p_0 - p_2\\p_2\end{pmatrix}$$

$$\varphi_N'(1) = p_0 + \frac{(N-1)\,p_2}{1-(p_1-p_2)} + \frac{(p_1-p_2)(1-(p_1-p_2)^{N-1})(\left(1-(p_1-p_2)\right)p_0 - p_2)}{(1-(p_1-p_2))^2}$$

Or

$$\varphi_N'(1) = p_0 + (N-1)p_2 + \gamma\frac{(N-1)(1-\gamma)-(1-\gamma^{N-1})}{(1-\gamma)^2}p_2 + \gamma\frac{1-\gamma^{N-1}}{1-\gamma}p_0$$

where $\gamma = p_1 - p_2$.

Let us calculate the first few N. For N=1, we have $\varphi_N'(1) = p_0$, which is trivial. For N=2,

$$\varphi_2'(1) = p_0 + p_2 + \gamma\frac{(1-\gamma)-(1-\gamma)}{(1-\gamma)^2}p_2 + \gamma\frac{1-\gamma}{1-\gamma}p_0$$

Or

$$\varphi_2'(1) = p_0 + p_2 + (p_1-p_2)p_0$$

For N=3,

$$\varphi_3'(1) = p_0 + 2p_2 + \gamma\frac{2(1-\gamma)-(1-\gamma^2)}{(1-\gamma)^2}p_2 + \gamma\frac{1-\gamma^2}{1-\gamma}p_0$$

Or

$$\varphi_3'(1) = p_0 + 2p_2 + \gamma\frac{2-(1+\gamma)}{(1-\gamma)}p_2 + \gamma(1+\gamma)p_0$$

Or

$$\varphi_3'(1) = p_0 + 2p_2 + \gamma p_2 + \gamma(1+\gamma)p_0$$

For N=4,

$$\varphi_4'(1) = p_0 + 3p_2 + \gamma \frac{3(1-\gamma) - (1-\gamma^3)}{(1-\gamma)^2} p_2 + \gamma \frac{1-\gamma^3}{1-\gamma} p_0$$

Or

$$\varphi_4'(1) = p_0 + 3p_2 + \gamma(2+\gamma)\, p_2 + \gamma(1+\gamma+\gamma^2)\, p_0$$

References

Viveros, R, Balasurbramanian, K, Balakrishnan, N (1994), “Binomial and negative Binomial Analogues Under Correlated Bernoulli Trials”, *The American Statistician*, **48,** 243-247.

Author

Yasunari Tosa is a Principal Software Engineer, Retica Systems Inc., Waltham, MA 02451 (email: ytosa@retica.com). The author thanks David Usher for helpful comments.